Structure of the space of reducible connections for Yang-Mills theories

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Abstract. *The geometrical structure of the gauge equivalence classes of reducible connections are investigated. The general procedure to determine the set of orbit types (strata) generated by the action of the gauge group on the space of gauge potentials is given. In the so obtained classification, a stratum, containing generically certain reducible connections, corresponds to a class of isomorphic subbundles given by an orbit of the structure and gauge group. The structure of every stratum is completely clarified. A nonmain stratum can be understood in terms of the main stratum corresponding to a stratification at the level of a subbundle.*

1. INTRODUCTION

Given the fact that the fundamental interactions in elementary particle physics are described by gauge theories, the study of gauge transformations acting on various fields entering in the theory is one of the first steps towards the understanding of the fundamental laws of nature. In the present work we study gauge transformations within a pure Yang-Mills theory, having in mind path integral quantization and anomalies. That is here we are not interested in the solutions of the classical equations of motion but in the full space of connections. In what follows, we would first like to explain how the problem

arises and make some considerations about its physical significance.

The Yang-Mills theory, defined on a principal bundle P of a compact, connected manifold M with structure group G, is determined by the action functional

$$(1) \qquad S : \mathcal{A} \to \mathbf{R}$$

where $\mathcal{A}$ is the space of gauge potentials or connections on P. The gauge transformations (the gauge group) $\mathcal{G}$ are acting non trivially on $\mathcal{A}$, but physics should not depend on them. This is guaranteed, at least at the classical level, by the $\mathcal{G}$-invariance of S. So eq. (1) is equivalent to

$$(2) \qquad S : \mathcal{M} \to \mathbf{R}$$

where $\mathcal{M}$ is the space of the gauge inequivalent connections, the orbit space of the $\mathcal{G}$ action on $\mathcal{A}$, $\mathcal{M} = \mathcal{A}/\mathcal{G}$. It is well-known that several physical properties of the theory are connected with the topological and geometrical structure of $\mathcal{M}$. Anomalies are such an example which has received a lot of attention in recent years [1]. Since $\mathcal{A}$ is an affine space, it is obvious that the nontrivial topological structure of $\mathcal{M}$ is the result of the gauge group action on $\mathcal{A}$ [2]. In general, this action leads to a nontrivial stratification on $\mathcal{A}$ [3]. The maximum symmetry group associated with a given connection $A_0 \in \mathcal{A}$, the stability group of A_0

$$(3) \qquad \mathcal{G}_{A_0} = \{ \psi \in \mathcal{G} \mid \psi^* A_0 = A_0 \}$$

in general, is not isomorphic to the corresponding stability group of a different connection A_1. That means $\mathcal{G}_{A_0} \not\cong \mathcal{G}_{A_1}$ for $A_0 \neq A_1$ is possible. The stability group $J := \mathcal{G}_{A_0}$ is isomorphic to a subgroup of the structure group G. The connections which belong to a given stratum of orbit type (J) are given by

$$(4) \qquad \mathcal{A}^{(J)} := \{ A \in \mathcal{A} \mid \mathcal{G}_A = \psi \circ J \circ \psi^{-1} \text{ with } \psi \in \mathcal{G} \} \ .$$

If we denote by $\mathcal{S}$ the set of orbit types which appear in $\mathcal{A}$ by the action of the gauge group $\mathcal{G}$, we obtain the stratification on $\mathcal{A}$:

$$(5) \qquad \mathcal{A} = \bigcup_{(J) \in \mathcal{S}} \mathcal{A}^{(J)}$$

$$(6) \qquad \mathcal{M} = \bigcup_{(J) \in \mathcal{S}} \mathcal{M}^{(J)}$$

S is a countable set [3] and $A \in \mathcal{A}^{(J)}$ can be considered as the connections with the symmetry type (J).

Except for the main stratum it is not at all clear as yet what role the stratification plays in physics. One of the main purposes of this paper is to clarify those features of the stratification which must be known for any further progress towards such an understanding. In [4] it was mentioned that an internal symmetry (J) leads to a conservation law. So we may ask whether the stratification of $\mathcal{A}$ tells us which conservation laws can occur in a Yang-Mills theory. One may further investigate the connection between stratification and anomalies [5]. Certainly, connections not belonging to the main stratum are reducible and are of special interest whenever a mechanism of the reduction of the structure group is involved. One may also investigate the connection between stratification and spontaneous symmetry breaking. Reducible connections are also needed in determining the low energy structure of superstring theories [6] and play a special role in topological quantum field theory [7].

In this paper we present the solution of two problems concerning the stratification. We make use of the results of [3] but go beyond them, taking into account the ideas of G-Theory [8]. Firstly, with given data the principal bundle $P(M, G)$, dim $(M) \geq 2$, we show how to obtain the set $S = \{(J)\}$ of orbit types which enter in the stratification generated by the gauge group $\mathcal{G}$ on the space of connections $\mathcal{A}$. This result was proven in [3] by associating a given connection A with the holonomy bundle $Q_A(M, H_A)$, using the isomorphism $\mathcal{G}_A \cong Z_G(H_A)$ (see below). Here we associate $\mathcal{A}^I$, as defined in eq. (7), with the maximal bundle $Q_J(M, H_J)$ and show how to obtain the set S. Secondly, the structure of a fixed stratum $\mathcal{A}^{(J)}$, called the orbit bundle, is analyzed [1]: we first consider the standard principal bundle $\mathcal{A}^J(\mathcal{M}^{(J)}, N(J)/J)$ of $\mathcal{A}^{(J)}$ [2] connected to the stability group J

$$(7) \qquad \mathcal{A}^J := \{A \in \mathcal{A} \mid \mathcal{G}_A = J\},$$

and which has structure group $N(J)/J$ with $N(J)$ the normalizer of J in $\mathcal{G}$. In addition, for a fixed point $p_0 \in P$, we consider the holonomy bundle of A $Q_A = Q_A(M, H_A)$ and the subgroup $H_J := Z_G(Z_G(H_A)) = Z_G(J)$, with $Z_G(H_A)$ the centralizer of the holonomy group H_A in G. Extending the structure group of Q_A to H_J we obtain a new principal bundle $Q_J = Q_J(M, H_J)$. The subbundle Q_J allows to reduce all connections $A \in \mathcal{A}^J$ to Q_J and to obtain the set $\tilde{\mathcal{A}}_{Q_J}$ as a subset of the main stratum in $\mathcal{A}_{Q_J}$ (the space of connections in Q_J). Then, showing that $N(J)/J$

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[1] This is particularly relevant for the non main orbit type strata.

[2] This can be shown by applying the slice theorem proven in [3] and the results of [8].

is isomorphic to $\overline{\mathcal{G}}_{Q_J} = \mathcal{G}_{Q_J}/C(H_J)$ where $\mathcal{G}_{Q_J}$ is the gauge group in Q_J and $C(H_J)$ the center of H_J, we obtain the isomorphism

$$(8) \qquad \mathcal{A}^{(J)} \cong \widetilde{\mathcal{A}}_{Q_J} \times_{\overline{\mathcal{G}}_{Q_J}} \mathcal{G}/J ,$$

This result is of course independent of the choice of the representative J in (J). It implies further that we have the isomorphism:

$$(9) \qquad \mathcal{M}^{(J)} \cong \widetilde{\mathcal{A}}_{Q_J}/\overline{\mathcal{G}}_{Q_J} .$$

The plan of the paper is as follows: In sect. 2 we prepare for our main result and classify the set S of orbit types determined by the stratification. Some of these results were also obtained in an unpublished work [3]. In sect. 3 we analyse the structure of a given orbit bundle and derive as our main result the above isomorphisms. Sect. 4 presents some conclusions.

2. THE CLASSIFICATION OF THE ORBIT TYPES OF THE G-ACTION ON $\mathcal{A}$

The aim of this section is to obtain a characterization of the orbit type (J) in a way which allows to determine the set S of orbit types which appear in the stratification of the space $\mathcal{A}$ by the gauge group $\mathcal{G}$, as given in eq. (5).

2.1 The maximal subbundle related to $\mathcal{A}^J$

Looking for a characteristic property of the connections in the orbit bundle $\mathcal{A}^{(J)}$, it seems useful to restrict oneself first to the associated standard principle bundle $\mathcal{A}^J$ whose elements correspond to the same fixed stability group J. Furthermore, it is well-known [9] that for a $p_0 \in P$ and $A \in \mathcal{A}^J$, the stability group J is connected with the holonomy group $H_A(p_0)$ by

$$(10) \qquad J \cong Z_G\left(H_A(p_0)\right) .$$

That means that the maximal symmetry group J of A is connected with the reducibility property of A. Unfortunately, this property is A-dependent, as different connections in $\mathcal{A}^J$ may have different holonomy groups and different holonomy bundles. There exists, however, another group, the maximal group associated to J, H_J and another subbundle of P, the maximal subbundle Q_J which is relevant to the standard principal bundle $\mathcal{A}^J$ itself. Therefore, as we are going to show, the Q_J can be used to characterize $\mathcal{A}^J$ and consequently also to characterize the orbit bundle $\mathcal{A}^{(J)}$. We start with a few definitions.

For A with stability group J, the holonomy bundle at $p_0 \in P$ with holonomy group H_A is given by $Q_A = Q_A(M, H_A)_{p_0}$. Then we define the *maximal group* H_J associated with H_A by

(11) $$H_J := Z_G\left(Z_G(H_A)\right)$$

and the *maximal subbundle* in P associated with Q_A by

(12) $$Q_J := Q_A \cdot H_J .$$

Furthermore, a subbundle $Q(M, H)$ of the principle bundle $P(M, G)$ is called *maximal*, if $Z_G(Z_G(H)) = H$. With these definitions the following proposition is valid:

PROPOSITION 1. *Every connection $A \in \mathcal{A}^J$ can be reduced uniquely to a connection on the subbundle Q_J, and Q_J is the maximal subbundle of P where $j \in J$ is constant.*

This can be seen as follows: For $A \in \mathcal{A}^J$ with stability group J and $j \in J$, we have $\psi_j^* A = j^{-1}Aj + j^{-1}dj = A$. $(^3)$ This means that the equivariant map

$$j : P \to G$$

is constant on horizontal curves in P. Therefore, the restriction of j on the holonomy bundle $Q_A(M, H_A)p_0$ $(^4)$ is also constant. On the other side, every equivariant mapping $j : P \to G$ whose restriction on Q_A is constant, belongs to the stability group of A $(j \in J)$. So we have for $j \in J$ and $p \in Q_A$, with $h \in H_A$

(13) $$j(ph) = h^{-1}j(p)h = j(p)$$

and it follows that

(14) $$j \in J \Leftrightarrow j : Q_A \to Z_G(H_A) \text{ constant .}$$

In addition, we have

(15) $$j(pg) = g^{-1}j(p)g = j(p)$$

$(^3)$ We make the identification between the vertical automorphisms on P, $\psi_f \in \mathrm{Aut}_V(P) = \mathcal{G}$ and the equivariant mappings $C(P, G) = \{f : P \to G \text{ equivariant}\}$ given by $\psi_f(p) = p \cdot f(p)$.

$(^4)$ Since all our considerations are related to some fixed $p_0 \in P$, we in general do not denote this dependence explicitly.

for $p \in Q_J$ and $g \in Z_G(Z_G(H_A)) = H_J$. That means that j is constant on the subbundle Q_J of P given by $Q_J = Q_A \cdot H_J$ with structure group H_J. Q_J is the maximal subbundle of P on which j can be constant. The same considerations apply also to another connection $A' \in \mathcal{A}^J$ from which follows that also the holonomy bundle $Q_{A'}$ is contained in the maximal subbundle Q_J. That means that for the holonomy group H_A of any $A \in \mathcal{A}^J$, $H_A \leq H_J$ is valid.

The reduction theorem of connections [10] guarantees then that every connection A with stability group J is reducible to a connection on the subbundle Q_J.   ■

The possible difference between the holonomy group H_A and the structure group H_J of the maximal subbundle Q_J is illustrated in the following example with $G = SU(3)$:

$$
\begin{array}{cccc}
 & H_A & J & H_J \\[4pt]
 & SU(2) & U(1) & U(2) \\[4pt]
(16) & U(1) \times U(1) & U(1) \times U(1) & U(1) \times U(1) \\[4pt]
 & U(1) & U(2) & U(1) \\[4pt]
 & \mathbf{1} & SU(3) & \mathbf{Z}_3
\end{array}
$$

The choice of the maximal subbundle Q_J gives rise to the isomorphism given by

$$
(17) \qquad
\begin{aligned}
\beta : J &\to Z_G(H_J) \\
j &\mapsto z = j(q) \in Z_G(H_J) \ .
\end{aligned}
$$

Note that β depends also on the choice p_0 since we have by construction $H_J = H_J(p_0)$ and $Q_J = Q_J(p_0)$. For a $p = p_0 g$ with $g \in G$ we have

$$
Q_J(p) = Q_J(p_0) \cdot g
$$

and

$$
Z_G(H_J(p)) = g^{-1} Z_G(H_J(p_0)) \, g \ .
$$

At this place, we would like to remark that, in constrast to the holonomy group, the maximal group H_J is always a closed subgroup of G. Therefore the reduced subbundle Q_J is equipped with the induced topology.

2.2 The independence on the choice of a representative in the orbit type

We have now to extend our considerations from $\mathcal{A}^J$ to $\mathcal{A}^{(J)}$. That means we have to show the independence of our results on the specific choice of the representative J

in (J). As already mentioned, the stratification of $\mathcal{A}$ is classified by the set of the orbit types $S = \{(J)\}$. The stratum $\mathcal{A}^{(J)}$ which is given by the orbit type (J), is represented by

$$\mathcal{A}^{(J)} = \mathcal{A}^J \times_{N(J)/J} \mathcal{G}/J$$

which is independent of the choice $J \in (J)$. For $J, J' \in (J)$ with $J' = \psi_f J \psi_f^{-1}$ for some $\psi_f \in \mathcal{G}$, we have $\mathcal{A}^J = \psi_f^* \mathcal{A}^{J'}$ which means

$$(18) \qquad J \overset{\psi_f}{\sim} J' \Leftrightarrow \mathcal{A}^J \overset{\psi_f}{\cong} \mathcal{A}^{J'} .$$

Now the question arises which is the connection between Q_J and $Q_{J'}$. The answer is given by the next proposition:

PROPOSITION 2. *If the stability groups J and J' are $\mathcal{G}$-conjugate with $J' = \psi_f \circ J \circ \psi_f^{-1}$ for some $\psi_f \in \mathcal{G}$, then there exist maximal subbundles corresponding to J and J' so that $Q_{J'} = \psi_f(Q_J)$.*

In order to prove this, we would first like to show a stronger formulation of the above proposition: under the same assumptions as above, there exists a $g \in G$ so that

$$(19) \qquad Q_{J'} = \psi_f(Q_J) \cdot g .$$

For $A, A' \in \mathcal{A}^{(J)}$ with stabilizer the groups J and J' respectively, and

$$(20) \qquad J' = f J f^{-1} \qquad \text{with} \qquad f \in \mathcal{G}$$

we consider the maximal subbundles associated to J and J', Q_J and $Q_{J'}$, respectively. We may now ask how Q_J and $Q_{J'}$ are related to each other. Taking $j \in J$ and $j' \in J'$ with

$$(21) \qquad j|_{Q_J} = z \qquad \text{and} \qquad j'|_{Q_{J'}} = z'$$

we first show that there exists a g so that

$$(22) \qquad z = g z' g^{-1} .$$

For $p' \in Q_{J'}, p \in Q_J$, we can have a g so that

$$(23) \qquad \psi_f^{-1}(p') \cdot g^{-1} = p \in Q_J .$$

From (21) we have

(24) $j(p) = z$ and $j'(p') = z'$

and from (20), (21), (23) and (24) we obtain

(25) $z = j(p) = j\left(\psi_f^{-1}(p')\,g^{-1}\right) = g\,j\left(\psi_f^{-1}(p')\right)\,g^{-1}$

and

(26) $z' = j'(p') = f(p')\,j(p')\,f(p')^{-1} = j\left(p' \cdot f(p')^{-1}\right) = j\left(\psi_f^{-1}(p')\right)$

so that indeed

(27) $z = g\,z'\,g^{-1}$.

Now, since we know that given $j|_{Q_J} = z$ and $j'|_{Q_{J'}} = z'$, there exists a $g \in G$ with $z = gz'g^{-1}$, we have for every $p' \in Q_{J'}$ a $p := \psi_f^{-1}(p') \cdot g^{-1}$ with

(28) $j(p) = j\left(\psi_f^{-1}(p') \cdot g^{-1}\right) = g\,j\left(\psi_f^{-1}(p')\right)\,g^{-1} = g\,j'(p')\,g^{-1}$.

So from the maximal property of $Q_{J'}$ it follows that $p \in Q_J$, so we have shown that

(29) $\psi_f^{-1}\left(Q_{J'}\right) \cdot g^{-1} \subset Q_J$ and $Q_{J'} \subset \psi_f\left(Q_J\right) \cdot g$.

Similarly, starting with $p \in Q_J$, we also obtain

(30) $\psi_f\left(Q_J\right) \cdot g \subset Q_{J'}$

and we have shown that

(31) $Q_{J'} = \psi_f\left(Q_J\right) \cdot g$.

Using now the fact that $Q_{J'} \cdot g^{-1} =: Q'_{J'}$ is also a maximal subbundle, we have

(32) $Q'_{J'} = \psi_f\left(Q_J\right)$

which is precisely the conjecture of the proposition.   ∎

It is easy to see that from $Q_{J'} = \psi_f(Q_J)g$ it follows that $J' = \psi_f\,J\,\psi_f^{-1}$.

2.3 The classification

Having in mind the results of the previous subsections, we can contruct a relation between the stratification of $\mathcal{A}$ eq. (5) on the one side, and some specific reduced sub-bundles of P on the other side. The interest focuses on the strata other than the main stratum $\overline{\mathcal{A}} := \mathcal{A}^{C(G)}$.

We consider the set $\mathcal{P}$ of reduced bundles Q with structure group $H < G$ so that $H = Z_G(Z_G(H))$, (this means that Q is maximal), and the property that if Q is not connected and H' is the structure group of a connected component, then there exists no maximal subgroup $\tilde{H} < G$ with $H' \leq \tilde{H} < H$ (5). Then we define on $\mathcal{P}$, the set of maximal subbundles of P with the above restriction, the following equivalence relation:

$$(33) \qquad Q' \sim Q \Leftrightarrow \exists \psi_f \in \mathcal{G} \quad \text{and} \quad g \in G \quad \text{so that} \quad Q' = \psi_f(Q)g \ .$$

We can now define the set $\mathcal{T} := \mathcal{P}/\sim$ which is bijective to S via the mapping.

$$(34) \qquad \begin{aligned} S &\to \mathcal{T} \\ (J) &\mapsto (Q_J) \ . \end{aligned}$$

So we have expressed the stratification of $\mathcal{A}$ by classes of reduced bundles in P . That means that the stratification is given by group-theoretical and topological informations. As one expects, there may be topological obstructions against the existence of reduced bundles [11]. The set $S \cong \mathcal{T}$ itself is a countable set [12, 13].

In order to demonstrate the relation between S and $\mathcal{T}$, we study the following example, where $S \cong \mathcal{T}$. We consider the trivial principal bundle $P = S^2 \times SU(2)$. The possible maximal subbundles are P itself and Q_n for $n \in \mathbb{N}$ with

$$(35) \qquad \begin{aligned} Q_0 = S^2 \times \mathbf{Z}_2, \qquad Q_1 = S^2 \times U(1), \qquad Q_2 = S^3 \\ \text{and} \qquad Q_n = S^3/\mathbf{Z}_{n-1} \qquad \text{for} \quad n \geq 3 \ . \end{aligned}$$

So we have

$$(36) \qquad \mathcal{T} = \left\{ P, \{Q_n\}_{n \in \mathbb{N}} \right\} \ .$$

The stratification of $\mathcal{A}$ is determined by $\mathcal{T}$:

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(5) Without this restriction there is no corresponding holonomy bundle.

The stability groups are given by

(37) $J_n := \left\{ j \mid j : Q_n \to Z_{SU(2)}(H_n) \quad \text{constant} \right\}$

where $H_0 = \mathbf{Z}_2$, $H_n \cong U(1)$ for $n \geq 1$' and

(38) $Z_{SU(2)}(H_0) = SU(2), \quad Z_{SU(2)}(H_n) \cong U(1)$

and

(39) $\mathcal{A} = \mathcal{A}^{(\mathbf{Z}_2)} \bigcup_{n \in \mathbf{N}} \mathcal{A}^{(J_n)}$.

$\mathcal{A}^{(\mathbf{Z}_2)}$ is the main stratum.

3. THE STRUCTURE OF THE ORBIT BUNDLES $\mathcal{A}^{(J)}$

As already mentioned in the previous section, the standard principal bundle $\mathcal{A}^J$, associated to the orbit bundle $\mathcal{A}^{(J)}$, contains some information about the reducibility of its elements but this information is not complete. In this section we would like to give the precise characterization of $\mathcal{A}^J$ which is related to the reducibility properties of its elements. This leads to a principal bundle isomorphism between $\mathcal{A}^J$ and a subset $\widetilde{\mathcal{A}}_{Q_J}$ of the main orbit bundle $\overline{\mathcal{A}}_{Q_J}$ in the space of connections $\mathcal{A}_{Q_J}$ of the maximal bundle Q_J .

3.1 The isomorphism between $\mathcal{A}^J$ and $\widetilde{\mathcal{A}}_{Q_J}$

In this subsection we give the precise definition of the space $\widetilde{\mathcal{A}}_{Q_J}$ and in addition prove the bijection between $\mathcal{A}^J$ and $\widetilde{\mathcal{A}}_{Q_J}$.

Let $Q_J = Q_J(M, H_J)$ be a maximal subbundle of P ,associated to the stability group $J < \mathcal{G}$, $\mathcal{A}_{Q_J}$ the space of connections on Q_J and

(40) $\beta : J \to G$

the homomorphism induced by Q_J from the stability group into the structure group. The space $\widetilde{\mathcal{A}}_{Q_J}$ is then defined by

(41) $\widetilde{\mathcal{A}}_{Q_J} := \left\{ A \in \mathcal{A}_{Q_J} \mid Z_G(H_A) = \beta(J) \right\}$

with H_A the holonomy group of A . From this definition it follows that

$$\mathcal{A}^{irr}_{Q_J} \subset \widetilde{\mathcal{A}}_{Q_J} \subset \overline{\mathcal{A}}_{Q_J} \subset \mathcal{A}_{Q_J}$$

where $\overline{\mathcal{A}}_{Q_J}$ is the main orbit bundle in Q_J and $\mathcal{A}_{Q_J}^{\text{irr}}$ are the irreducible connections in Q_J. The following proposition is valid:

PROPOSITION 3. *The connections in* P *with stability group* J *are isomorphic to* $\widetilde{\mathcal{A}}_{Q_J}$:

$$\mathcal{A}^J \cong \widetilde{\mathcal{A}}_{Q_J} .$$

In order to see this, let

(42)
$$\alpha : \mathcal{A}^J \to \mathcal{A}_{Q_J}$$
$$A \mapsto A^{\text{red}}$$

be the mapping which associates to every connection $A \in \mathcal{A}^J$ its reduced connection $A^{\text{red}} \in Q_J$. Every connection from $\mathcal{A}^J$ is uniquely reducible to a connection on Q_J, hence this mapping is injective. Since A and A^{red} have the same holonomy group, it follows that

(43)
$$Z_G(H_{A^{\text{red}}}) = \beta(J)$$

so we have

$$\alpha : \mathcal{A}^J \to \widetilde{\mathcal{A}}_{Q_J} .$$

Note also that the stability group of A^{red} is given by $\beta(J) \cap H_J = C(H_J)$.

In order to show the surjectivity, we take $\omega \in \widetilde{\mathcal{A}}_{Q_J}$. So there exists uniquely an element $A \in \mathcal{A}$ with $\omega = A^{\text{red}} = \alpha(A)$ since for the holonomy groups we have again $H_{A^{\text{red}}} = H_A$ and $Z_G(H_A) = Z_G(H_\omega) = \beta(J)$. So J is the stability group of A and the surjectivity of α is proven. ∎

In general $\widetilde{\mathcal{A}}_{Q_J}$ is a subspace of the main orbit bundle $\overline{\mathcal{A}}_{Q_J}$. This situation is related to the fact that the stratification contains only some incomplete information about reducibility. Taking as an example $P = M \times SU(3)$, M contractible, we can see that indeed $\widetilde{\mathcal{A}}_{Q_J} \neq \overline{\mathcal{A}}_{Q_J}$ may occur. In the following table we exhibit the various stability and maximal subgroups of $SU(3)$.

	J	H_J
1	$\mathbf{Z}_3$	$SU(3)$
2	$U_Y(1)$	$U(2)$
3	$U_Y(1) \times U_T(1)$	$U_Y(1) \times U_T(1)$
4	$U(2)$	$U_Y(1)$
5	$SU(3)$	$\mathbf{Z}_3$

Here $U_Y(1)$ denotes the $U(1)$ of the Hypercharge and $U_T(1)$ the $U(1)$ subgroup generated by the 3-component of isospin. Corresponding to case 3, a connection ω on the maximal subbundle $Q_{J_3}(M, U_Y(1) \times U_T(1))$ which is reducible to a connection on the subbundle $Q_{J_3}(M, U_Y(1))$ is considered. Since H_{J_3} is an Abelian group, we have $\mathcal{A}_{Q_{J_3}} = \overline{\mathcal{A}}_{Q_{J_3}}$. This ω can be considered as a connection A on P (with $A^{\mathrm{red}} = \omega$). Its stability group is given by $Z_G(H_A)$. Since ω is reducible to $H_A = U_Y(1)$, we have $Z_G(U(1)) = U(2)$ which means that it belongs to $\mathcal{A}^{J_4} \cong \widetilde{\mathcal{A}}_{Q_{J_4}}$. So we have

$$(44) \qquad \widetilde{\mathcal{A}}_{Q_{J_3}} \subset \overline{\mathcal{A}}_{Q_{J_3}}.$$

In contrast in case 2 we have $\widetilde{\mathcal{A}}_{Q_{J_2}} = \overline{\mathcal{A}}_{Q_{J_2}}$.

3.2 The group isomorphism between $N(J)/J$ and $\mathcal{G}_{Q_J}/C(H_J)$

As we already know, $\widetilde{\mathcal{A}}_{Q_J}$ is a subspace of $\overline{\mathcal{A}}_{Q_J}$, the main orbit bundle in $\mathcal{A}_{Q_J}$. Since the group $\overline{\mathcal{G}}_{Q_J}$ acts on $\widetilde{\mathcal{A}}_{Q_J}$ freely, $\widetilde{\mathcal{A}}_{Q_J}$ is also a principal bundle with fibers isomorphic to $\mathcal{G}_{Q_J}/C(H_J) =: \overline{\mathcal{G}}_{Q_J}$. We may therefore suspect the bijection shown in the last proposition between $\mathcal{A}^J$ and $\widetilde{\mathcal{A}}_{Q_J}$ to be a principal bundle isomorphism. In order to prove that this is indeed the case, we first show the group isomorphism between $N(J)/J$ and $\mathcal{G}_{Q_J}/C(H_J)$, and then the bijection between $\mathcal{A}^{(J)}/\mathcal{G}$ and $\widetilde{\mathcal{A}}_{Q_J}/\overline{\mathcal{G}}_{Q_J}$.

PROPOSITION 4. *There is the group isomorphism* $N(J)/J \cong \mathcal{G}_{Q_J}/C(H_J)$.

The main step to prove this is to realize that $N(J)$ can be written as $N(J) = \mathcal{G}_{Q_J} \cdot J$. We first define the group

$$(45) \qquad \widetilde{N}(J) := \{f : Q_J \to N_G(Z_G(H_J)) \mid f \in \mathcal{G}_{Q_J} \quad \text{equivariant}\}.$$

For $n \in N(J)$ and $j, j' \in J$, we have

$$(46) \qquad n j n^{-1} = j'$$

and for $q \in Q_J$ we have

$$(47) \qquad n(q) \, j(q) \, n(q)^{-1} = j'(q)$$

with $j(q)$, $j'(q) \in Z_G(H_J)$. So $n(p) \in N_G(Z_G(H_J))$ and $N(J) < \widetilde{N}(J)$.

Now we determine the condition for $n \in \widetilde{N}(J)$ to be element of $N(J)$. For q and $p \in Q_J$, since $j(q) = j(p)$ and $j'(q) = j'(p)$, we have

$$(48) \qquad n(p) j(p) n(p)^{-1} = n(q) j(p) n(q)^{-1}$$

and

$$
\begin{aligned}
(49) \quad j(p) &= n(p)^{-1} n(q) j(p) n(q)^{-1} n(p) \\
&= n(p)^{-1} n(q) j(p) (n(p)^{-1} n(q))^{-1}
\end{aligned}
$$

and $n(p)^{-1} n(q) \in Z_G(Z_G(H_J)) = H_J \; \forall p, q \in Q_J$.

So we can write for a fixed $p \in Q_J$

$$
(50) \qquad n(q) = n(p) \left(n(p)^{-1} n(q) \right)
$$

and we define

$$
(51) \qquad
\begin{aligned}
&\varphi_1 : Q_J \to H_J \\
&q \mapsto \varphi_1(q) := n(p)^{-1} n(q) .
\end{aligned}
$$

So we have $n(q) = n(p) \varphi_1(q)$. Since $Z_G(H_J)$ is closed in G, we can write

$$
(52) \qquad N_G(Z_G(H_J)) = Z_G(H_J) \cdot H_J
$$

$$
(53) \qquad n(p) = z(p) h(p)
$$

with $z(p) = z(q) \in Z_G(H_J)$ and $h(p) \in H_J$ we can write

$$
(54) \qquad n(q) = z(q) \cdot \varphi(q)
$$

with

$$
(55) \qquad
\begin{aligned}
&\varphi : Q_J \to H_J \\
&q \mapsto \varphi(q) := h(p) \varphi_1(q)
\end{aligned}
$$

equivariant. Since $zh = hz$, φ is indeed equivariant, which follows from eq. (54):

$$
(56) \qquad z\varphi(qh) = n(qh) = h^{-1} n(q) h = h^{-1} z \varphi(q) h = z h^{-1} \varphi(q) h
$$

$\varphi \in \mathcal{G}_{Q_J}$ and we have shown that

$$
(57) \qquad N(J) = \mathcal{G}_{Q_J} \cdot J .
$$

From this, it follows that

$$
(58) \qquad N(J)/J \cong \mathcal{G}_{Q_J} \cdot J/J \cong \mathcal{G}_{Q_J}/J \cap \mathcal{G}_{Q_J} \cong \mathcal{G}_{Q_J}/C(H_J)
$$

which completes our proof of the above proposition and we have the group isomorphism

$$
(59) \qquad
\begin{aligned}
\tilde{\beta} : \quad N(J)/J &\to \mathcal{G}_{Q_J}/C(H_J) \\
[n] &\mapsto [\varphi]
\end{aligned}
$$

∎

3.3 The principal bundle isomorphism between $\mathcal{A}^J(\mathcal{M}^{(J)}, \mathcal{G}/J)$ and $\widetilde{\mathcal{A}}_{Q_J}(\widetilde{\mathcal{M}}_{Q_J}, \mathcal{G}_{Q_J}/C(H_J))$

PROPOSITION 5. *There is the isomorphism* $\mathcal{A}^{(J)}/\mathcal{G} \cong \widetilde{\mathcal{A}}_{Q_J}/\overline{\mathcal{G}}_{Q_J}$.

In order to prove this, we first define the mapping

$$
(60) \qquad
\begin{aligned}
\gamma: \quad \widetilde{\mathcal{A}}_{Q_J}/\overline{\mathcal{G}}_{Q_J} \quad &\rightarrow \quad \mathcal{A}^{(J)}/\mathcal{G} \\
[\omega] \quad &\mapsto \quad \gamma([\omega]) := [\hat{\alpha}(\omega)]
\end{aligned}
$$

with $\hat{\alpha} = \alpha^{-1}$, the inverse of α (see proposition 3) and where $\omega \in \widetilde{\mathcal{A}}_{Q_J}$ and $\hat{\alpha}(\omega) =: \hat{\omega} \in \mathcal{A}^{(J)}$. We first have to show that γ is well defined: taking $[\omega_1]$ and $[\omega]$ with $\omega_1 = \psi_\varphi^* \omega$ and $\psi_\varphi \in \mathcal{G}_{Q_J}$, we use $\hat{\alpha}(\psi_\varphi^* \omega) = \psi_\varphi^*(\hat{\alpha}(\omega))$. This is the equivariance of the map $\hat{\alpha}$ under the gauge group. On the right hand side of the last equation $\psi_\varphi \in \mathcal{G}_{Q_J}$ is uniquely extended to P . We have

$$
(61) \qquad [\hat{\alpha}(\omega_1)] = [\hat{\alpha}(\psi_\varphi^* \omega)] = [\psi_\varphi^*(\hat{\alpha}(\omega))] = [\hat{\alpha}(\omega)]
$$

so that γ is well defined.

For the injectivity of γ we have to show that from ω and $\theta \in \widetilde{\mathcal{A}}_{Q_J}$, with

$$
(62) \qquad \gamma([\omega]) = \gamma([\theta]) = [A]
$$

$[\omega] = [\theta]$ is valid. From eq. (62) we have $\hat{\alpha}(\omega) = A = \hat{\omega} \in \mathcal{A}^J$ and

$$
(63) \qquad \hat{\alpha}(\theta) = \psi_f^* A = \psi_f^* \hat{\omega} \in \mathcal{A}^J \text{ with } \psi_f \in \mathcal{G} .
$$

Since $\hat{\omega}, \psi_f^* \hat{\omega} \in \mathcal{A}^J$ for $j \in J$, we have

$$
(64) \qquad j^* \circ \psi_f^* \hat{\omega} = \psi_f^* \hat{\omega} = \psi_f^* \circ j^* \hat{\omega}
$$

which means that $\psi_f \in N(J)$ for which we can write $\psi_f = j_1 \circ \psi_\varphi$ with $\psi_\varphi \in \mathcal{G}_{Q_J}$ and $j_1 \in J$, as was shown in the previous proposition. So we have

$$
(65) \qquad \theta = \alpha(\psi_f^* \hat{\omega}) = \alpha(\psi_\varphi^* \hat{\omega}) = \psi_\varphi^*(\alpha(\hat{\omega})) = \psi_\varphi^* \omega
$$

which shows the injectivity of γ .

For the surjectivity we have to show that every $[B] \in \mathcal{A}^{(J)}/\mathcal{G}$ is the image of some element of $\widetilde{\mathcal{A}}_{Q_J}/\overline{\mathcal{G}}_{Q_J}$. If B has J' as its stability group, it is possible to find an $f \in \mathcal{G}$, so that $J' = f^{-1} J f$. Then $\psi_f^* B = A$ has J as its stability group and

$\alpha(A) = A^{\text{red}} \in \tilde{A}_{Q_J}$. So we have $\hat{\alpha}(A^{\text{red}}) = A$ and $\gamma([A^{\text{red}}]) = [A] = [B]$ which shows also the surjectivity of γ . ∎

From the isomorphism between $\mathcal{A}^J$ and $\tilde{A}_{Q_J}$, given in subsection 3.1, and from the last two propositions, the principle bundle isomorphism

$$(66) \qquad \mathcal{A}^J(\mathcal{M}^{(J)}, N(J)/J) \cong \tilde{A}_{Q_J}(\widetilde{\mathcal{M}}_{Q_J}, \mathcal{G}_{Q_J}/C(H_J))$$

can be shown immediately. Given the isomorphisms

$$\begin{array}{ccc}
\alpha: & \mathcal{A}^J & \to & \tilde{A}_{Q_J} \\
& \downarrow & & \downarrow \\
\gamma: & \mathcal{M}^{(J)} & \to & \widetilde{\mathcal{M}}_{Q_J}
\end{array}$$

(67)

and

$$\begin{array}{cccc}
\tilde{\beta}: & N(J)/J & \to & \mathcal{G}_{Q_J}/C(H_J) \\
& [n] & \mapsto & [\varphi] = \tilde{\beta}[n]
\end{array}$$

we have only to show that

$$(68) \qquad \alpha\left([n]^*A\right) = \tilde{\beta}([n])^* \alpha(A)$$

(with $A \in \mathcal{A}^J$ and $[n] \in N(J)/J$) is valid. This is easy to see since we have the equivariance of the map α (used already in the proof of proposition 5):

$$(69) \qquad n^*A = [n]^*A = \varphi^*A = [\varphi]^*A$$

and

$$(70) \qquad \alpha\left[[n]^*\right] = \alpha\left([\varphi]^*A\right) = [\varphi]^*\alpha(A) = \tilde{\beta}([n])^* \alpha(A) .$$

With the above principle bundle isomorphism, we can express the orbit bundle $\mathcal{A}^{(J)}$ by

$$(71) \qquad \mathcal{A}^{(J)} \cong \tilde{A}_{Q_J} \times_{\mathcal{G}_{Q_J}/C(H_J)} \mathcal{G}/J .$$

4. CONCLUSION

In this paper we have studied the nonfree action of the gauge group $\mathcal{G}$ on the space of gauge potentials $\mathcal{A}$ on a principle bundle $P(M, G)$. We have analysed the structure

of the so obtained stratification on $\mathcal{A}$ and in particular the structure of a fixed stratum $\mathcal{A}^{(J)}$ in the general case, where $\mathcal{A}^{(J)}$ is not the main stratum. We were able to show that every nonmain stratum $\mathcal{A}^{(J)}$ can be understood in terms of the main stratum which corresponds to the stratification of the space of connections on a subbundle $Q_J(M, H_J)$ of P and to give the general procedure which allows to determine the set of orbit types $S = (J)$ generated by the action of the group $\mathcal{G}$ on the space $\mathcal{A}$.

As yet, it is not at all clear what role the stratification plays in physics. It may well be that the stratification is an important clue in the quantization of Yang-Mills theories whose role may have been completely overlooked. Our analysis constitutes the first step in clarifying this question. In addition to this, the stratification seems to be closely connected with the question of conservation laws, anomalies and even the mechanism of spontaneous symmetry breaking.

In order to recognize the structure of the stratum $\mathcal{A}^{(J)}$, we observe that its elements, the reducible connections, have in general different holonomy groups and consequently correspond to different holonomy bundles. The same is true even if we restrict ourselves to a standard principle bundle $\mathcal{A}^J$, the connections with fixed stability group J $(\mathcal{A}^J \subset \mathcal{A}^{(J)})$. Nevertheless, as we have shown, there exists a subbundle of P, the maximal subbundle Q_J, and every connection in $\mathcal{A}^J$ is reducible to a connection on Q_J. This «rough» reducibility is the property which characterizes the elements of $\mathcal{A}^J$. It is interesting to note the reason why this happens and that a direct construction of the maximal subbundle Q_J may be realized and obtained by the methods of G-theory discussed in [13]. Here we would like only to point out that the reason for the correspondence between $\mathcal{A}^J$ and Q_J is that both are standard principal bundles of the $\mathcal{G} \times G$-action on $\mathcal{A}$ and P respectively, with essentially the same stability group isomorphic to J. This explains also the fact that the stratum $\mathcal{A}^{(J)}$ corresponds to a $\mathcal{G} \times G$ orbit of Q_J as it was shown in our second proposition (Ch. 2.2). From this point of view the classification of the stratification (Ch. 2.3) also seems plausible. For a given stratum $\mathcal{A}^{(J)}$ in particular, by the use of its equivariant properties, we were able to prove the following: the orbit spaces $\mathcal{A}^{(J)}/\mathcal{G}$, a part of the configuration space of Yang-Mills theories, is isomorphic to the $\tilde{\mathcal{A}}_{Q_J}/\overline{\mathcal{G}}_{Q_J}$. $\tilde{\mathcal{A}}_{Q_J}$ is a specific subset (defined in (41)), of the main stratum of connections on the maximal subbundle Q_J corresponding to the stratum (J) (in some cases it is isomorphic to it). Together with the group isomorphism $N(J)/J \cong \overline{\mathcal{G}}_{Q_J}$ we have the principal bundle isomorphism of $N(J)/J \to \mathcal{A}^J \to \mathcal{A}^{(J)}/\mathcal{G}$ and $\overline{\mathcal{G}}_{Q_J} \to \tilde{\mathcal{A}}_{Q_J} \to \tilde{\mathcal{A}}_{Q_J}/\overline{\mathcal{G}}_{Q_J}$ with $\overline{\mathcal{G}}_{Q_J} = \mathcal{G}_{Q_J}/C(H_J)$. $\mathcal{A}^J$, the connections with a fixed stabilizer J, is the underlying principal bundle of the homogeneous bundle $\mathcal{A}^{(J)} \cong \mathcal{A}^J \times_{N(J)/J} \mathcal{G}/J$.

With the above isomorphisms we can represent the nonmain orbit bundle $\mathcal{A}^{(J)}$ in terms of (a part of) the main orbit bundle $\tilde{\mathcal{A}}_{Q_J}$:

(71) $$\mathcal{A}^{(J)} \cong \tilde{\mathcal{A}}_{Q_J} \times_{\overline{\mathcal{G}}_{Q_J}} \mathcal{G}/J \, .$$

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